

Rescuing a Castaway

Keywords: plane geometry, Pythagorean theorem, perpendicular bisector

A plane is searching the open sea for a castaway who has a device on their raft that emits a distress signal. The device has only a limited range. While flying over the sea, the crew picks up the signal, but after a short time, it is lost. The pilot turns the aircraft around, and they manage to receive the signal again, though only briefly.

The trajectory of the entire flight, including the direction of travel and the points where the signal was picked up (points A_1 and A_2) and lost (points B_1 and B_2) is shown on the map.

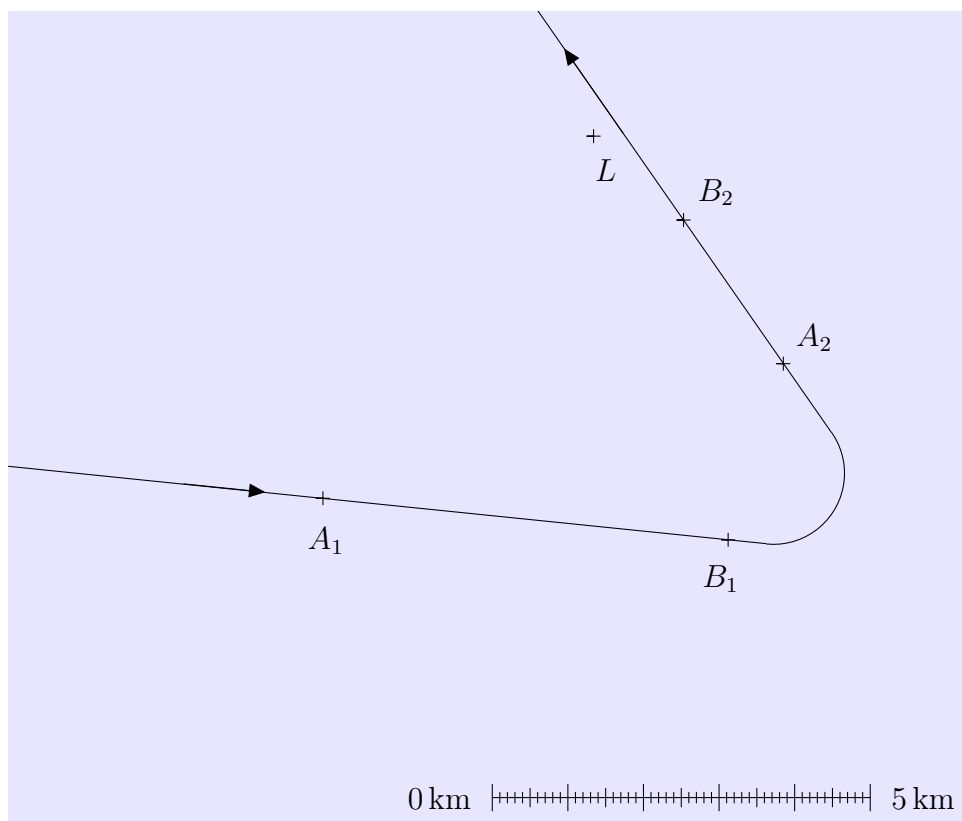


Figure 1: Flight trajectory of the aircraft

During both periods when the crew received the signal, the aircraft maintained a constant altitude. Between points B_1 and A_2 , the aircraft descended by 500 m.

Exercise 1. Use a geometric construction on the map to determine the position X of the castaway.

Solution. The limited range of the castaway's distress signal defines a hemisphere above the sea surface, with the center located at the castaway's position. Horizontal cross-sections of this hemisphere are circles that appear on the map as concentric circles centered at point X .

Since the aircraft maintained a constant altitude between points A_1 and B_1 , the segment A_1B_1 forms

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a chord of a certain circle k_1 with center at point X . Therefore, point X must lie on the perpendicular bisector of segment A_1B_1 . For the same reason, point X must also lie on the perpendicular bisector of segment A_2B_2 , since this segment is a chord of another circle k_2 centered at X .

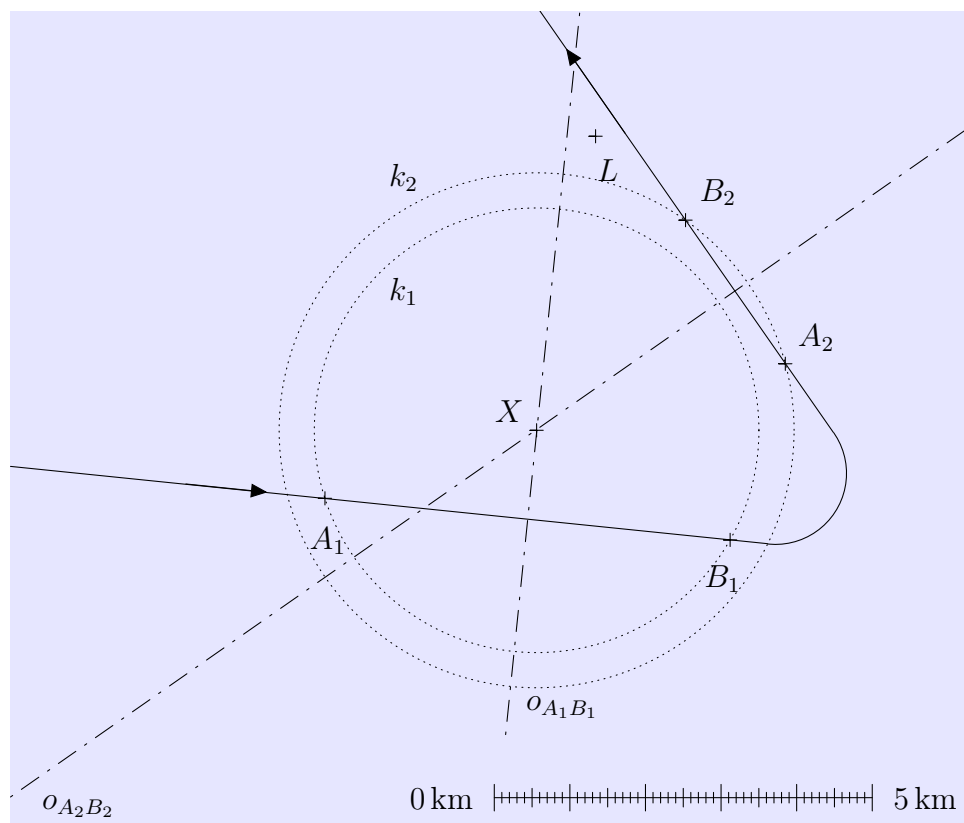


Figure 2: Solution to Exercise 1

Exercise 2. There is a cargo ship in the area (position L). Can it also receive the castaway's distress signal, or is it too far away?

- Transfer the lengths of segments LX , A_1X , and A_2X from the solution to Exercise 1 to the scale provided. Using these distances (rounded to the nearest smallest unit of the scale), solve the problem numerically.
- Using the construction from Exercise 1, solve the problem again—this time relying solely on geometric constructions.

Solution.

- To solve the problem, we need to determine the range of the castaway's device, which is the radius r of the hemisphere mentioned in the solution to the previous exercise. By transferring the segments A_1X and A_2X to the scale and rounding their lengths to the nearest tenth of a kilometer, we get $|A_1X| \doteq 2.9$ km and $|A_2X| \doteq 3.4$ km. These lengths are clearly the radii r_1 and r_2 of the circles k_1 and k_2 .

Let us consider a projection of the hemisphere in which the circles k_1 and k_2 appear as parallel segments

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K_1L_1 and K_2L_2 , such that they share the same perpendicular bisector o , their lengths are $2r_1$ and $2r_2$ respectively, and the vertical distance between them is 0.5 km. Let S be the center of the hemisphere, S_1 the midpoint of segment K_1L_1 , and S_2 the midpoint of K_2L_2 . See the figure below, where the sea level is also marked as a straight line h for clarity.

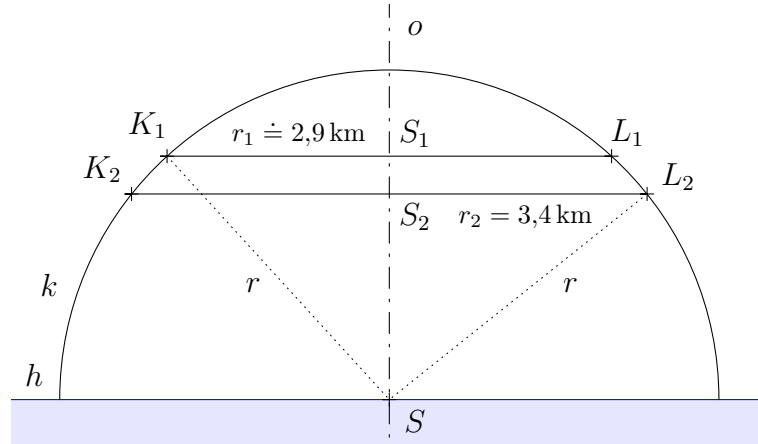


Figure 3: Auxiliary projection of the hemisphere used in solving Exercise 2a)

Using the Pythagorean theorem on triangles SS_1K_1 and SS_2L_2 , we obtain the following equations:

$$\begin{aligned} r^2 &= r_1^2 + |SS_1|^2 \\ r^2 &= r_2^2 + |SS_2|^2. \end{aligned}$$

We also know that $|SS_1| = |SS_2| + 0.5$. Substituting this into the first equation and comparing both expressions, we obtain a linear equation with a single unknown $|SS_2|$, which we solve as follows:

$$\begin{aligned} r_2^2 + |SS_2|^2 &= r_1^2 + (|SS_2| + 0.5)^2 \\ |SS_2| &= r_2^2 - r_1^2 - 0.25 \end{aligned}$$

Solving for r using the second equation and substituting, we get:

$$r = \sqrt{r_2^2 + (r_2^2 - r_1^2 - 0.25)^2} \doteq 4.5 \text{ km.}$$

The distance from the ship to the castaway is given by the length of segment LX . By transferring this segment to the scale, we find $|LX| \doteq 4.0$ km, which is less than the range r of the castaway's signal. Therefore, the ship can receive the signal.

- b) To construct a geometric solution to the problem (i.e., to determine the radius r of the hemisphere), we use the same auxiliary projection of the hemisphere as in Exercise 2a. The center of the hemisphere S is the intersection of the common perpendicular bisector o of segments K_1L_1 and K_2L_2 with the perpendicular bisector of segment L_1L_2 , since L_1L_2 is a chord of the semicircle outline k . The desired radius r is, for example, the length of segment SK_1 — see the figure.

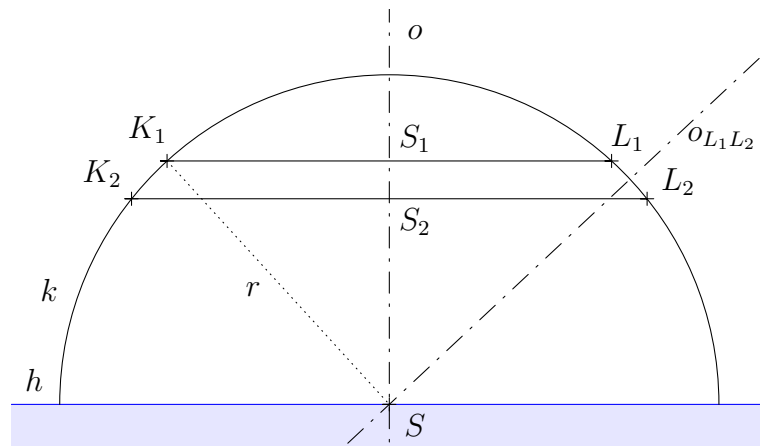


Figure 4: Auxiliary projection of the hemisphere used in solving Exercise 2b)

To carry out the construction, we transfer the distances r_1 and r_2 from the solution to Exercise 1 (recall that $r_1 = |A_1X|$ and $r_2 = |A_2X|$), as well as the distance between the centers of the circles, $|S_1S_2| = d_{0.5}$, where $d_{0.5}$ is the map distance corresponding to 0.5 km, taken from the scale.

The projection of the hemisphere on the map is bounded by a circle l with center at point X and radius r , which we transfer from the auxiliary projection. Once this circle is drawn, it is clear that the ship is within range of the distress signal.

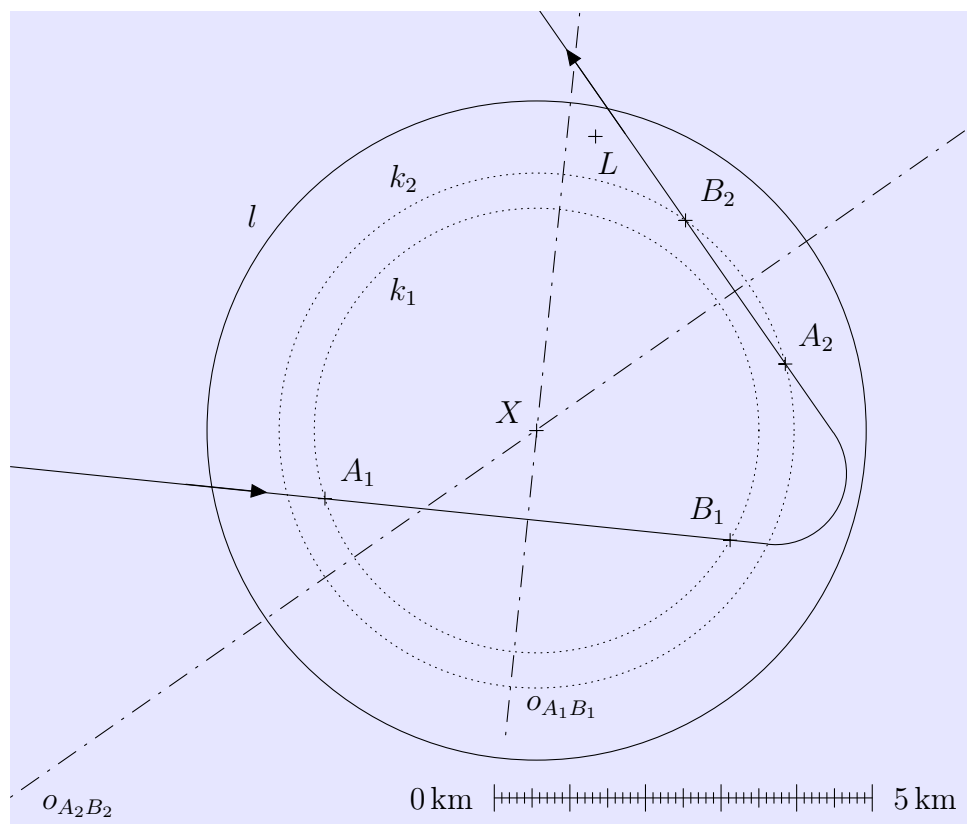


Figure 5: Solution to Exercise 2b)