

The Curious Warehouse Manager

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When we solve purely mathematical problems, we get exact results. However, when we use mathematics to solve problems in the world around us, we rarely achieve absolute precision in the answer. Approximation is sometimes the result of a simplification of the real situation in our minds. Sometimes the input data are approximated (e.g. we can only measure lengths or time with limited accuracy) or an absolutely exact result is realistically unattainable and must be rounded off.

Rounding to a given number of significant digits is often used in practice (and in the following problems). We round a positive real number r to n significant digits as follows:

- We express r in the form $a \cdot 10^b$, where $a \in \mathbb{R}, a \in \langle 1, 10 \rangle$ and $b \in \mathbb{Z}$, and then we round the number a to $n - 1$ decimal places according to the standard rules for rounding.
- E.g. numbers $r = 31.258\,16$ and $s = 0.023\,123\,6$ we round to four valid digits as follows:

$$\begin{aligned} r &= 31.258\,16 = 3.125\,816 \cdot 10^1 \doteq 3.126 \cdot 10^1 = 31.26 \\ s &= 0.023\,123\,6 = 2.312\,36 \cdot 10^{-2} \doteq 2.312 \cdot 10^{-2} = 0.023\,12. \end{aligned}$$

Notably, rounding input data can have surprising consequences for the accuracy of the result, for example when solving equations, as we will see in the following series of problems.

Exercise 1. The manager of the pharmaceutical warehouse received an invoice for two types of ordered vaccines. A total of 401,950 CZK was paid for the delivery of 597 packages of Ixodinum vaccines against encephalitis and 386 packages of Nopolio vaccines against polio. However, during initial inspection, 86 packages of Ixodinum vaccine and 19 packages of Nopolio vaccine were found to be expired and had to be returned. A total of 39,600 CZK was refunded for the expired medicines. Out of curiosity, the manager wants to calculate the purchase price of one package of both vaccines. However, he does not have a calculator or a mobile phone on hand, so he settles for an approximate solution. He rounds all the figures he knows to one significant digit before calculating. How much will his result differ from the actual purchase price? For both types of vaccines, determine the absolute difference between the calculated and actual prices, as well as the relative error expressed as a percentage.

Solution. First, let's solve the problem without rounding. Let x be the price per package of Ixodine and y be the price per package of Nopolio. The information in the assignment leads to a system of two linear equations with two unknowns

$$\begin{aligned} 597x + 386y &= 401,950 \\ 86x + 19y &= 39,600 \end{aligned}$$

whose solution gives us the real purchase price of a package of Ixodin vaccine 350 CZK and of a package of Nopolio vaccine 500 CZK.

After rounding the coefficients to one significant digit, we solve the system

$$\begin{aligned} 600x' + 400y' &= 400\,000 \\ 90x' + 20y' &= 40\,000. \end{aligned}$$

The solution is the pair $x' = \frac{1,000}{3} \doteq 333$ and $y' = 500$. Now we have the actual price of the medicine and the estimate of the price by the warehouse manager. Let's also calculate the relative error in the

Results matter!

price of the medicine due to rounding. Relative error is the absolute error (absolute value of price difference) divided by the actual price per package. We summarize the results in the table:

vaccine	actual price	price estimate	relative error
Ixodinum	350 Kč	333 Kč	$\frac{350-333}{350} = 4,9\%$
Nopolio	500 Kč	500 Kč	$\frac{500-500}{500} = 0\%$

Exercise 2. After a few months, another delivery arrived at the warehouse, namely 504 packages of Antiflu vaccines against influenza and 81 packages of Kontradift vaccines against diphtheria. 198,900 CZK was paid for this delivery. During initial inspection, 98 packages of Antiflu and 18 packages of Kontradift were found to be expired. A total of 40,700 CZK was refunded. The warehouse manager repeated his procedure and calculated the approximate purchase price of the two drugs off the top of his head. This time, however, he was surprised. What was the reason for his surprise and how much did his result differ from the actual prices?

Solution. We will solve the problem in the same way as before, this time we will denote x the price of one package of Antiflu and y the price of one package of Kontradift. The real prices are the solution of the system

$$\begin{aligned} 504x + 81y &= 198,900 \\ 98x + 18y &= 40,700 \end{aligned}$$

where we get $x = 250$ and $y = 900$.

When rounding the coefficients, we solve the system

$$\begin{aligned} 500x' + 80y' &= 200,000 \\ 100x' + 20y' &= 40,000, \end{aligned}$$

whose solution is $x' = 400$ and $y' = 0$. From the solution of the warehouse manager, it seems that the second vaccine was delivered to the warehouse free of charge, while it is actually almost four times more expensive than the first one. We calculate the relative error and enter all the values in the table again:

vaccine	actual price	price estimate	relative error
Antiflu	250 Kč	400 Kč	$\frac{400-250}{250} = 60\%$
Kontradift	900 Kč	0 Kč	$\frac{900-0}{900} = 100\%$

Exercise 3. Graphically represent the systems of equations from the previous two problems using appropriate software. Explain the difference in the accuracy of the results of the two exercises by comparing their graphs.

Solution. Let p_1 , p_2 (or q_1 , q_2) be the lines given by the equations of the system with unrounded coefficients in Exercise 1 (or Exercise 2), namely

$$\begin{aligned} p_1 : 597x + 386y &= 401,950 \\ p_2 : 86x + 19y &= 39,600 \\ q_1 : 504x + 81y &= 198,900 \\ q_2 : 98x + 18y &= 40,700. \end{aligned}$$

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Let us denote the lines given by the corresponding equations with rounded coefficients by p'_1 , p'_2 , q'_1 and q'_2 and further denote the points $P \in p_1 \cap p_2$, $P' \in p'_1 \cap p'_2$, $Q \in q_1 \cap q_2$ and $Q' \in q'_1 \cap q'_2$. A graphical representation of the pair of systems for each problem separately is shown in following figure.

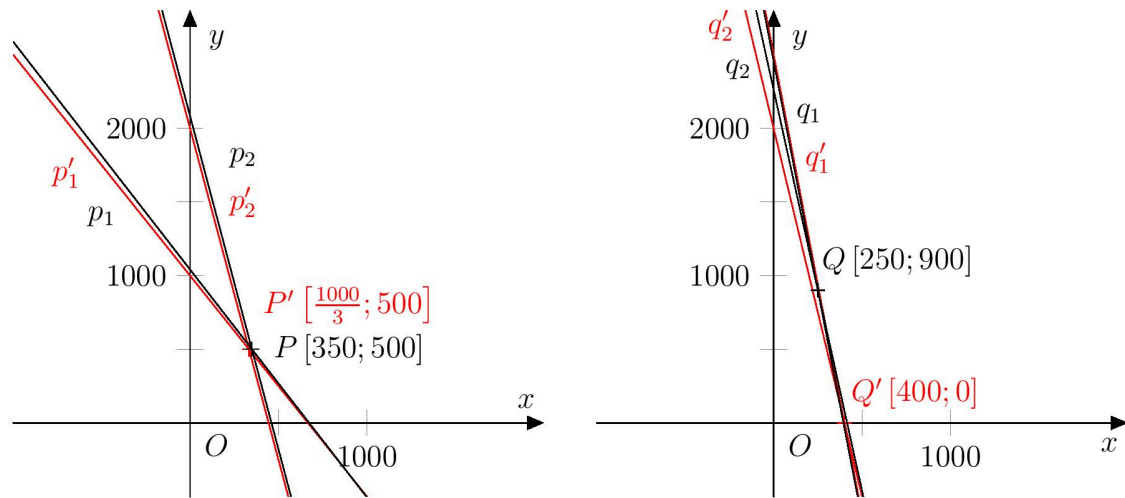


Figure 1: Graphic representation of the systems

By comparing the two graphical representations, it can be seen that in the case of Exercise 2, the pair of lines q_1 and q_2 are almost parallel. When rounding the coefficients of the equation, the position of the lines relative to the coordinate system generally changes and the position of the intersection changes as well. The change in the position of the intersection is much greater for lines that are almost parallel. The figure also shows why the second coordinate of the intercept (i.e., the price of the vaccine Contradift) will be much more affected by rounding in the second problem. Because of the slope of the lines q_1 and q_2 , a small change in the x coordinate of the intersection would mean a large change in its y coordinate.

Literature

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