

Optimal Ticket Price

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The carousel owner wants to set the ticket price for his attraction so that his profit is maximized. For simplicity, he sets the entry fee in multiples of 10 CZK. He knows that at the current price of 50 CZK, he can expect about 600 visitors per day. From previous years, he estimates that if the ticket price increases by 10 CZK, about 50 fewer visitors will come per day. Conversely, if the ticket price drops by 10 CZK, 50 more visitors will come. Additionally, to increase visitor interest, every third visitor will receive cotton candy worth 30 CZK at the carousel owner's expense.

Since the son of the carousel owner happens to be studying mathematics, he decides to help his father with this problem.

Exercise 1. After some thought, the son wrote the formula

$$y = (50 + 10x) \cdot (600 - 50x) - \frac{600 - 50x}{3} \cdot 30.$$

Interpret the individual parts of this formula.

Solution. The following table explains the meaning of each part of the formula:

Part of the Formula	Meaning
$10x$	change in ticket price by $x \cdot 10$ CZK
$50 + 10x$	new ticket price
$600 - 50x$	estimated number of visitors after price change
$(50 + 10x)(600 - 50x)$	daily ticket earnings
$\frac{600 - 50x}{3}$	number of visitors receiving cotton candy
$\frac{600 - 50x}{3} \cdot 30$	daily cost of cotton candy
y	total daily profit

Exercise 2. If we consider the previous formula as a function of the variable x , what type of function is it and what does its graph look like?

Solution. By simplifying the right side of the formula, we convert it to the form

$$y = -500x^2 + 4\,000x + 24\,000.$$

This is a quadratic function whose graph is a parabola. Due to the negative coefficient of the quadratic term, this parabola is concave down.

Exercise 3. At what ticket price would the owner make the maximum profit?

Solution. Our task is to determine the maximum of the function

$$f(x) = -500x^2 + 4\,000x + 24\,000.$$

We know that the graph of the function f intersects the x -axis at points corresponding to the roots of the quadratic polynomial. At the midpoint of the line segment joining these points, we can find the

Results matter!

maximum of the function we are looking for (due to the symmetry of the parabola). Therefore, we determine the roots of the quadratic polynomial.

$$-500x^2 + 4000x + 24000 = 0$$

$$x^2 - 8x - 48 = 0$$

$$(x - 12)(x + 4) = 0$$

The solutions of the equation are the roots $x_1 = 12$ and $x_2 = -4$, so the maximum of the function f is at the point $x_{max} = \frac{12-4}{2} = 4$. We have thus increased the original ticket price by 40 to a new price of 90 CZK to achieve the maximum profit.

Exercise 4. How much is the maximum profit greater than the profit at the original ticket price?

Solution. The answer to this problem is the difference $f(4) - f(0)$. By substituting both values into the function f , we get $f(4) = 32000$ and $f(0) = 24000$. The difference between the two amounts is 8000 CZK. The graph of the function f with the values $f(4)$ and $f(0)$ marked is shown in the figure:

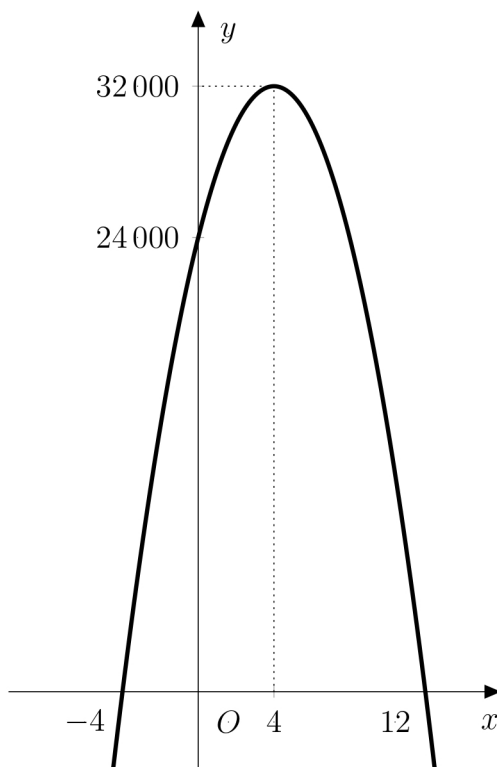


Figure 1: Profit Function Graph

Exercise 5. When creating the formula, the son simplified one aspect of the problem. Do you know which one? Could you solve the problem without this simplification? Compare your model with the one created by the son of the carousel owner.

Solution. The son's formula is accurate only if the number of visitors per day $600 - 50x$ is divisible by three. If it is not divisible by three, then the number of visitors who received the gift is the lower integer part of the number $\frac{600-50x}{3}$, denoted as $\lfloor \frac{600-50x}{3} \rfloor$.

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The function that takes as input a real number x and gives as output the greatest integer m such that $m \leq x$ is called *the floor function*. For example, $\lfloor \frac{5}{2} \rfloor = 2$, $\lfloor \pi \rfloor = 3$ nebo $\lfloor -8,3 \rfloor = -9$.

Now, let's compare the previously used function f with the new function g , which uses the lower integer part:

$$g(x) = (50 + 10x) \cdot (600 - 50x) - \left\lfloor \frac{600 - 50x}{3} \right\rfloor \cdot 30.$$

$x_{max} = 4$. We will compare the results around the already determined maximum, $x_{max} = 4$. For x , we will substitute values from the interval $\langle 3,5; 4,5 \rangle$, considering values for which both the ticket price and the number of visitors are integers. It can be observed that it is sufficient to ensure that the ticket price is an integer, which will occur for values of x with at most one decimal place. The functions values can be summarized in a table (it is advantageous to use software like MS Excel for calculations):

x	3,5	3,6	3,7	3,8	3,9	4,0	4,1	4,2	4,3	4,4	4,5
$f(x)$	31 875	31 920	31 955	31 980	31 995	32 000	31 995	31 980	31 955	31 920	31 875
$g(x)$	31 895	31 920	31 965	32 000	31 995	32 010	32 015	31 980	31 965	31 940	31 875

The daily profit determined by the function g is always greater than or equal to the daily profit determined by the function f . Also, the maximum of the function g is at a different point, specifically 4,1, which corresponds to a ticket price of 91 CZK and an estimated daily attendance of 395 people.

We can observe the following facts:

- The differences between the profits are very small, within fractions of the price of a single ticket. It doesn't matter whether the ticket costs 90 or 91 CZK, because the only person who comes in extra (or doesn't come in) compared to the estimates, reliably erases any difference between the profits of the two models.
- While we determine the maximum of the function f by a direct and relatively simple calculation, we had to determine the maximum of the function g by substituting all of the permissible values (from a relatively narrow interval).

Therefore, despite the simplification, using the function f to calculate the maximum profit is both sufficient and more convenient.