

Crossing the river

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The ferryman must get from his dock on one river bank to the other dock, which is located on the opposite bank, 500 m downstream. The river between the docks is straight and 100 m wide. The speed of the current is 2 m/s. We also know that the ferryman's boat moves at a speed of 12 km/h relative to the water.

Exercise 1. The ferryman wants to travel directly from one dock to the other. He can achieve this by angling the boat across the river and maintaining this direction. By what angle should the ferryman turn his boat from the direct path to travel straight to the other dock?

Solution. Let us denote by A and B the starting and destination point. Also, denote by P the foot of the perpendicular line from point B to the opposite bank of the river. Next, mark the vectors of the current's velocity, the boat's velocity relative to the water, and the resulting velocity of the boat (boat's velocity relative to the river's bottom) all with their initial points at point A and endpoints as follows (see the figure):

- $\overrightarrow{AM}$ is the vector of the current's velocity.
- $\overrightarrow{AL}$ is the vector of the boat's velocity relative to the water (not including current).
- $\overrightarrow{AK}$ is the vector of the resulting velocity of the boat relative to the river's bottom (current + boat velocity).

Since $K \in AB$, our task is to determine the angle KAL , which we denote by α . By the vector addition rule, the quadrilateral $MALK$ is a parallelogram.

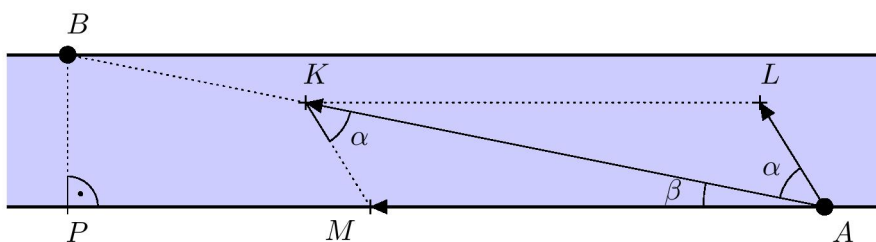


Figure 1: Crossing the river.

Further, let's denote $|\angle MAK| = \beta$. Then $|\angle MAL| = \alpha + \beta$. It is given that $|BP| = 100$ m and $|AP| = 500$ m. By using the tangent function in the right-angled triangle PAB we get $\beta = \arctan \frac{1}{5} \approx 11^\circ 19'$.

Since a parallelogram is bisected by its diagonal into two congruent triangles, it holds that $|\angle AKM| = |\angle KAL| = \alpha$. To calculate α , we use the law of sines for the triangle AKM . After converting the side lengths to a common unit (km per hour in our solution), we get specifically $|KM| = 12$ km/h and $|AM| = 2 \cdot 3.6 = 7.2$ km/h. We now express α from the law of sines:

$$\frac{|KM|}{\sin \beta} = \frac{|AM|}{\sin \alpha}$$

$$\sin \alpha = \frac{|AM|}{|KM|} \cdot \sin \beta \quad \Rightarrow \quad \alpha = \arcsin \left(\frac{|AM|}{|KM|} \cdot \sin \beta \right)$$

After substituting the values, we obtain $\alpha \approx 6^\circ 45'$. Thus, the ferryman has to turn his boat by approximately 7° to the right from the direct course to the destination.

Exercise 2. If the digital clock on the boat shows 11:00 at the moment of departure (without showing seconds), what time will the clock show at the moment the boat arrives at the other dock?

Solution. Let's first determine the distance between the two docks using the Pythagorean theorem in the right-angled triangle ABP :

$$\begin{aligned} |AB| &= \sqrt{|AP|^2 + |BP|^2} \\ |AB| &= \sqrt{0.5^2 + 0.1^2} \text{ km} \\ |AB| &\approx 0.51 \text{ km.} \end{aligned}$$

Now, we determine the magnitude of the resultant velocity of the boat relative to the river's bottom, which is equal to the length of the line segment AK . This, for example, can be determined using the law of cosines in triangle AKM . We denote the interior angle $|\angle AMK| = \gamma$, which measures $180^\circ - \alpha - \beta \approx 161^\circ 56'$. Then for $|AK|$ we can write:

$$\begin{aligned} |AK| &= \sqrt{|KM|^2 + |AM|^2 - 2 \cdot |KM| \cdot |AM| \cdot \cos \gamma} \\ |AK| &= \sqrt{12^2 + 2^2 - 2 \cdot 12 \cdot 2 \cdot \cos(161^\circ 56')} \text{ km/h} \\ |AK| &\approx 19.0 \text{ km/h} \end{aligned}$$

The boat travels a direct path of 0.51 km at an average speed of 19 km/h, which takes:

$$\begin{aligned} t &= \frac{\text{distance traveled}}{\text{average speed}} \\ t &= \frac{0.51}{19} \text{ h} \end{aligned}$$

which is approximately 97 seconds. Therefore, the clock will show 11:01 or 11:02 upon arrival at the target dock.